The work done on a system of particles (DRAFT).

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Resumo

There is a very common derivation of the work done by the forces on a system of point particles that is found in most books of Classical Mechanics (Goldstein, Saletan, Marion, Lemos) that seems to have missing information on the dependence of each function, here we try to present a more carefull derivation.

The point is that a dynamical system is always evolving in time, and we must consider in every integral, the position of every other particle when we are considering the movement of a single one. Suppose a system of N point particles, labeled by the index i, in a given time t, each one have a position $\vec{s}_i(t)$ and we know all the $\{\vec{s}_1(t), \vec{s}_2(t), ..., \vec{s}_N(t)\}$ with respect to some arbitrary origin of an inertial observer.
We would like to calculate, the work done by all the forces on this system, during a time period $t_A \leq t \leq t_B$. Let's first evaluate the total force acting on a particle in a given time, it is given by:

$$\boldsymbol{F}_i(t) = \left[\sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{f}_{i,j}(t) + \boldsymbol{F}_i^e(t)\right] \tag{1}$$

Where $\boldsymbol{F}_i^e$ is the external force acting on i, and $\boldsymbol{f}_{i,j}(t)$ is the internal force, on particle i due to interaction with particle j, at a given time. Of course this force must be a function of the positions of each particle, so let's write:

$$\boldsymbol{f}_{i,j}(t) = \boldsymbol{f}(\boldsymbol{s}_i - \boldsymbol{s}_j) \tag{2}$$

During this time interval, the total work done on a single particle by the forces acting on it is:

$$W_i = \int_{t_A}^{t_B} \boldsymbol{F}_i(t) \cdot \frac{d\boldsymbol{s}_i}{dt}(t)dt \tag{3}$$

Where $\boldsymbol{s}_i$ is the path traced by the particle, ultimately determined by the dynamical equtions. We can now sum over all particles, to obtain the total work, which is what we are trying to calculate:

$$W = \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} \int_{t_A}^{t_B} \boldsymbol{F}_i(t) \cdot \frac{d\boldsymbol{s}_i}{dt}(t)dt \tag{4}$$

Substituting the expression for the force:

$$W = \sum_{i=1}^{N} \int_{t_A}^{t_B} \left[\sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{f}_{i,j}(t) + \boldsymbol{F}_i^e(t) \right] \cdot \frac{d\boldsymbol{s}_i}{dt}(t)dt \tag{5}$$

Distributing the scalar product:

$$W = \sum_{i=1}^{N} \tag{6}$$